The Hammer-and-Nail Phenomenon in Mathematics Education

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"For a person with a hammer, everything looks like a nail" is a proverb that can be used to highlight the phenomenon that students tend to rely on familiar ideas as opposed to taking time to think about and analyse a problem. Presented in this theoretical paper is the usefulness of the hammer-and-nail metaphor, other related theoretical constructs, pedagogical causes of student impulsive behaviours, and pedagogical suggestions for addressing them.

The behaviour of "doing whatever first comes to mind … or diving into the first approach that comes to mind" (Watson & Mason, 2007, p. 307) is quite common among students when solving problems in mathematics. Consider, for example, a 9^{th} grader's response to the task of finding the value of b that would make the equation $(b - 1)(b + 4) = 0$ true. As shown in Figure 1, the student multiplied out the given factors and then re-factored the quadratic equation before using the zero-product property. This student's behaviour is reminiscent of an English proverb: *For a person with a hammer, everything looks like a nail*. Metaphorically, the student saw the factored-form equation as a "nail" that compelled him to use the multiplying-out-factors algorithm, commonly known as FOIL, as a "hammer" to obtain the general form.

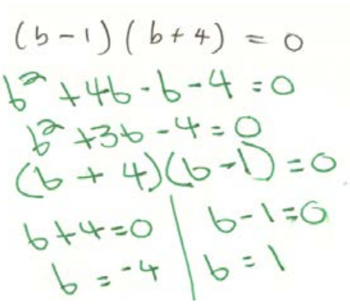

Figure 1. A 9^{th} grader's solution

This hammer-and-nail proverb can be used as a means to draw educators' attention to this phenomenon. In this paper, the following aspects of the hammer-and-nail metaphor are discussed: (a) its origin and its usefulness, (b) its manifestations among learners of mathematics, (c) related literature, (d) pedagogical causes, and (e) suggestions for teaching.

Origin and Usefulness of the Hammer-and-Nail Metaphor

The hammer-and-nail metaphor was first utilised in the context of conducting scientific research in the 1960's. Maslow (1966) acknowledged that "it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail" (p. 15-16). He distinguished between two types of researchers: *problem-centring* scientists who "choose to work as best they can with important problems" and *method-centring* scientists who restrict "themselves to doing only that which they can do elegantly with the techniques already available" (p. 16). Kaplan (1964) coined the phrase *the law of the instrument* to refer to the individual scientists' tendency to formulate a problem according to her or his expertise.

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Exposing the hammer-and-nail disposition can help researchers become aware of their tendency to be confined by the theoretical perspectives and methods they are most familiar with. In the context of solving mathematics problems, making the hammer-and-nail phenomenon explicit can help teachers and students become aware of their tendency to rely on algorithms they know as opposed to taking time to think and analyse a problem.

The author used the hammer-and-nail phenomenon as a metaphor to describe students' impulsivity in their problem solving at several professional development workshops for mathematics teachers. The participants generally related very well to this metaphor, especially when they experienced their own impulsivity through solving certain problems in a non-threatening environment. Several presentations on the hammer-and-nail phenomenon have been given at mathematics teacher conferences and have been well received by the audience.

Manifestations of Hammer-and-Nail Behaviours in Mathematics

The hammer-and-nail phenomenon is noticeable when an error is observed to arise from inappropriately applying a mathematical tool (hammer) to solve a mathematical problem (nail). Tools that are commonly used by students include formulas (e.g. quadratic formula), algorithms (e.g., setting up a proportion), strategies (e.g., identifying key words), and intuitions (e.g., multiplication makes bigger).

Treating formulas as "hammers" is common among students, especially when their understanding is superficial. For example, 52% of 307 pre-service K-8 teachers selected "d" as the answer for the question shown in Figure 2. These students interpreted the item as a speed problem (nail), read the distance from the graph (3000m) that corresponds to the 20 minutes, applied the $s = d/t$ formula (hammer), and obtained 150 by dividing 3000 by 20. Only 18% selected the correct answer "b" which is based on understanding speed as ratio of change in distance to change in time. Interestingly, 28% interpreted this task (nail) as a graph-reading activity (hammer) and chose 3000 meters as the answer.

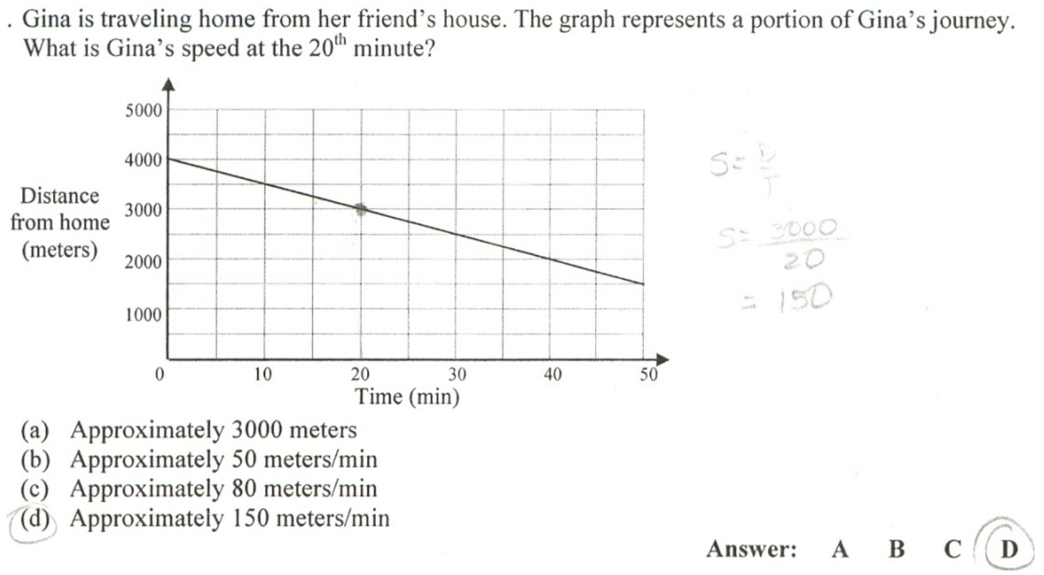


Figure 2. An inappropriate use of $s = d/t$ formula to find speed

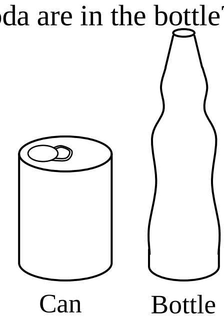
Student overgeneralization of proportionality in solving non-proportional missing-value problems can be interpreted as instantiations of the hammer-and-nail phenomenon. For example, 72.1% of Flemish fifth-graders applied a proportional strategy (hammer) to solve the following missing-value problem (nail): "*A group of 5 musicians plays a piece of music in 10 minutes. Another group of 35 musicians will play the same piece of music. How long will it take this group to play it?*" (Van Dooren, De Bock, Hessels, Janssens, & Vershaffel, 2004a, p. 390). The inappropriate use of proportional methods is found to be most prominent in Grade 5 where students in Belgium are taught extensively to reason proportionally. This overgeneralization is also common among pre-service teachers (Cramer, Post, & Currier, 1993; Lim, 2009; Monteiro, 2003).

The phenomenon of indiscriminately applying proportional strategies was observed in an exploratory study involving pre-service elementary and middle school teachers in a course on rational numbers and algebraic reasoning. After being taught ratios and proportions, these prospective teachers performed worse on all three non-proportional items in a post-test but performed better on all four direct-proportional items. A direct-proportional item and an inverse-proportional item are shown in 3a and 3b, respectively. We found that (a) the number of students who chose the correct answer "b" for the direct-proportional item increased from 64% to 78% (n = 118); (b) the number of students who chose the correct answer "a" for the inverse-proportional item, on the other hand, dropped from 55% to 42%; and (c) the number of students who chose the incorrect proportional answer "d" increased from 24% to 40%. These results suggest that when students are exposed to a particular solution strategy (hammer), they are more likely to use it even for situations where its use is inappropriate.

<u>Direct-Proportional Item</u>

The ratio of the amount of soda in the can to the amount of soda in the bottle is 4:3. There are 12 fluid ounces of soda in the can, how many fluid ounces of soda are in the bottle?

(a) 9 fluid ounces
(b) 11 fluid ounces
(c) 15 fluid ounces
(d) 16 fluid ounces
(e) None of the above



<u>Inverse-Proportional Item</u>

The ratio of the volume of a small glass to the volume of a large glass is 3:5. If it takes 15 small glasses to fill the container, how many large glasses does it take to fill the container?

(a) 9 glasses
(b) 13 glasses
(c) 17 glasses
(d) 25 glasses
(e) None of the above

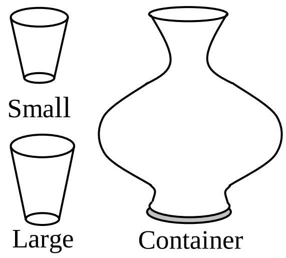


Figure 3. Two math test items used in a pilot study

In this paper, students' mistakes due to their familiarity with a particular idea or schema are considered manifestations of the hammer-and-nail phenomenon. It is important to note that the hammer-and-nail metaphor is just one of many viable ways of interpreting a situation. Other related perspectives are presented in the ensuing section.

Related Literature

Einstellung Effect. Students' tendency to approach a problem with conceptual tools that are familiar to them has been noted in the literature since the 1940s. In the famous water jar experiment (Luchins & Luchins, cited in NRC, 2000), after solving many problems using one approach subjects spontaneously used that approach to solve other problems that could have been more easily solved using a different approach. This phenomenon of solving a given problem in a fixated manner even when a better approach exists is called the *Einstellung effect*.

Spurious Correlation. In mathematics, student hammer-and-nail behaviours can be explained in terms of Ben-Zeev and Star's (2001) *spurious correlation*, which is a two-phase process consisting of: (a) conceiving an association between a problem feature and an algorithm for solving the problem, and (b) using the algorithm upon perceiving the same feature in another problem. In their study, students not only relied on surface-level features in solving problems but "also generate[d] and use[d] correlations between irrelevant surface-level features and solution strategies" (p. 272). Experienced students were also found to be susceptible to this tendency. Although the Einstellung effect is the scientific term for this phenomenon, the hammer-and-nail metaphor is less technical and more relatable to mathematics teachers.

Intuitive Rule. Tirosh and Stavy (1999) put forth a set of intuitive rules to account for many alternative conceptions in mathematics and science. For example, students who thought that the heavier the object the faster it falls are said to have relied on the "more A – more B" intuitive rule. Metaphorically, intuitive rules are the hammers that tend to influence students to perceive certain situations as nails.

Dual-System Theories. Several researchers in cognitive psychology have posited that there are two distinct cognitive systems of reasoning. Various names have been used to distinguish the two systems: implicit-explicit (Reber, 1993), associative and rule-based (Sloman, 1996), and System 1 and System 2 (Stanovich & West, 2000). According to Evans (2006), "System 1 processes are rapid, parallel and automatic in nature: only their final product is posted in consciousness" whereas "System 2 thinking is slow and sequential in nature and makes use of the central working memory system" (p. 454). Sloman (1996) points out that the two systems often work cooperatively despite having different goals and specializing in different kinds of tasks. At times, they may each try to generate a response. Although System 1 may have its response overridden by System 2, it "always has its opinion heard, and because of its speed and efficiency, often precedes and thus neutralizes the [System 2] response" (p. 15). Manifestations of the hammer-and-nail phenomenon may be attributed to the domination of System 1 over System 2.

Cognitive Tempo. On an individual-basis level, some students are more likely exhibit impulsive behaviours than other students. Such impulsivity can be considered a *cognitive style*–"a person's typical or habitual mode of problem solving, thinking, perceiving and remembering" (Riding & Indra, 1991, p. 194). In the field of psychology, the term *conceptual tempo* refers to a cognitive style that is along the impulsivity-reflectivity dimension (Kagan, 1965). Kagan, Rosman, Day, Albert, and Phillips (1964) constructed the *Matching Familiar Figures Test* to measure children's cognitive tempo. They classified a child who had an above-the-median response time and a below-the-median accuracy rate as having an impulsive style, and a child with a below-the-median response time and above-the-median accuracy rate as having a reflective style. Nietfeld and Bosma (2003) describe *impulsives* as "individuals who act without much forethought, are spontaneous, and take more risks in everyday activities" (p. 119) and *reflectives* as "more cautious, intent upon correctness or accuracy, and [who] take more time to ponder situations" (p. 119). Students classified as having an impulsive cognitive tempo are considered more likely to exhibit behaviours that are reminiscent of the hammer-and-nail phenomenon.

Impulsive Disposition. Lim, Morera, and Tchoshanov (2009) use the term *impulsive disposition* to refer to a tendency to proceed with an action that comes to mind without analysing the problem situation and without considering the relevance of the anticipated action to the problem situation. Impulsive disposition differs from impulsive tempo in that the latter is characterized by a fast but inaccurate response whereas the former is characterized by proceeding with an approach that comes to mind without checking its relevance, and not necessarily by how fast an approach comes to mind. Another difference is that impulsive tempo is a cognitive style that is rather stable across time and across situations (Messer & Brodzinsky, 1981), whereas impulsive disposition is a cognitive tendency which depending upon the circumstances may or may not result in an action. Impulsive disposition alludes to a mental tendency that can be modified under favourable learning conditions. In terms of the hammer-and-nail metaphor, students can learn to recognize situations where a particular idea is inappropriate from those where it is.

Knowing-to-Act in the Moment. Spontaneously responding to a situation with the first thing that comes to mind is not necessarily an undesirable way of thinking. Mathematicians often respond automatically to situations with which they are familiar. Mason and Spence (1999) describe this ability as *knowing-to act in the moment*. A person with this ability in relation to a particular tool is considered to have mastered the use of the tool, knows when not to use it, and consequently is able to avoid hammer-and-nail behaviours.

Pedagogical Causes of Hammer-and-Nail Behaviours

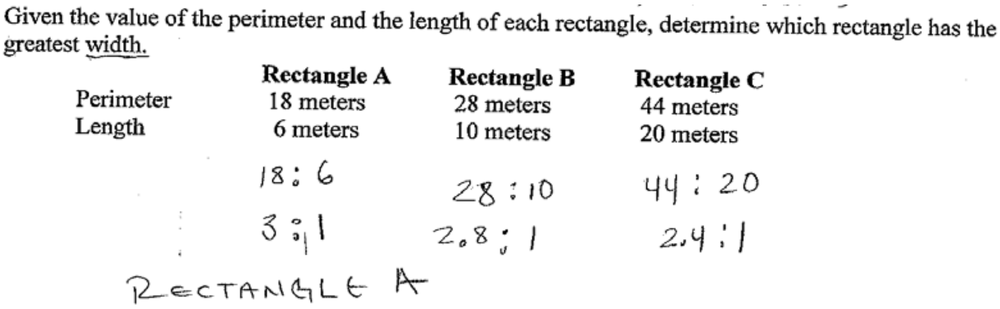
The traditional methods of teaching mathematics tend to foster hammer-and-nail behaviours. “The tradition has been to regard ‘mathematics’ as a set of rules for writing symbols on paper, and to regard the ‘teaching’ of mathematics as merely a matter of ‘telling’ students what to write and where to write it, together with supervising some considerable amount of drill and practice” (Davis, 1989, p. 159). Stigler and Hiebert (1999) characterized U.S. teaching as “learning terms and practicing procedures”, as opposed to “structured problem solving” (p. 27) in Japanese teaching. More than three-fifths of U.S. teachers described *skills* as the *main thing* that they wanted their students to learn; “they wanted the students to be able to perform a procedure, solve a particular kind of problem, and so on” (p. 89). Drill and practice of procedures without a conceptual focus will lead to superficial understanding and foster the System 1 mode of operation.

Mathematics curricula are typically organized sequentially, one chapter after another. Textbook problems are designed for students to practice the main ideas in a chapter. Such problems can be regarded as “nails” for students to practice with their newly acquired “hammer”. To help students remember certain facts or procedures, some teachers offer students learning aids such as acronyms (e.g., FOIL), schematic tools (e.g., ratio box to find the missing value from three given values), and key words (e.g., *altogether* means add and *share* means divide). Students typically learn such strategies by associating a feature with a procedure; these associations constitute the bases for hammer-and-nail behaviours. For example, problems that begin with “In a sports car race” tend to cue formulas such as $s = d/t$ or $d = rt$ (Hinsley, Hayes, and Simon, 1977). Such strategies suppress System 2 mode of reasoning because students are relieved from having to grapple with the mathematics they learn and from having to make sense of the problems they try to solve.

Suggestions for Addressing Hammer-and-Nail Behaviours

Addressing the hammer-and-nail phenomenon involves helping students develop a cognitive habit of maintaining *control* over their own mathematical thinking as they solve problems or learn mathematics. To be in control of a situation a student needs to: (a) be aware of his or her impulsive disposition, (b) stop relying solely on *instrumental understanding*—“rules without reason” (Skemp, 1976, p. 20), and (c) develop *relational understanding*—“knowing what to do and why” (p. 20). To help students become cognizant of their impulsive tendency, superficially-similar-but-structurally-different problems should be used intermittently in class as well as in examinations. To help students overcome their reliance on instrumental understanding, reasoning and sense making such as attendance to meaning of mathematical symbols (Harel, Fuller, & Rabin, 2009) should be emphasized in class. These instructional strategies were used in a study (Lim & Morera, 2010) that was conducted to explore the possibility of helping prospective K-8 teachers overcome their overgeneralization of proportionality within one semester.

Using Superficially-Similar-but-Structurally-Different Problems. After learning how to solve certain problems using a particular idea, students can work on a superficially-similar-but-structurally-different problem that can elicit a conceptual error and thereby allow students to realize the harmfulness of indiscriminately applying a newly learned idea. For example, when students have learned the use of ratio for comparing measures such as steepness of a line or "squareness" of a rectangle, the problem in Figure 4 can be posed. This problem offers students an opportunity to make a mistake, learn from their mistake, and become cognizant of their impulsive tendency. When used as classroom activities, superficially-similar-but-structurally-different problems allow teachers to emphasize quantitative reasoning, to foster attendance to meaning, and to extend student understanding by knowing when an idea is not applicable.


Given the value of the perimeter and the length of each rectangle, determine which rectangle has the greatest width.

	Rectangle A	Rectangle B	Rectangle C
Perimeter	18 meters	28 meters	44 meters
Length	6 meters	10 meters	20 meters
	18 : 6	28 : 10	44 : 20
	3 : 1	2.8 : 1	2.4 : 1

RECTANGLE A

Figure 4. A pre-service teacher's incorrect use of ratio to compare widths of rectangles

Emphasizing Reasoning and Sense Making. Awareness of one's impulsivity is useful but is not sufficient to ensure a correct solution: one needs to analyse and visualize the problem situation. For example, in solving a word problem, one needs to make sense of the problem by focusing on quantities—attributes that can be measured or amounts that can be counted—and understanding how those quantities are related, instead of identifying numbers and deciding which operations to use on those numbers. Thompson (1993) defines *quantitative reasoning* as "the analysis of a situation into a quantitative structure—a network of quantities and relationships" (p. 165). One way to foster quantitative reasoning is to challenge students to explain the quantitative structure underlying a problem situation. For example, when students set up a proportion $a/b = c/x$ to solve a missing-value problem (values of three quantities are given to find the value of the fourth quantity), they should provide "reasons in support of claims made about the structural relationships among four quantities" (Lamon, 2007, p. 638). Lim (2009) used non-proportional situations, in the context of burning candles, to emphasize the importance of analysing the problem situation, determining the co-varying quantities, and identifying the invariant relationship. The quantities depicted in a non-proportional missing-value problem may be related by an invariant difference ($a – b = c – x$), an invariant sum ($a + b = c + x$), or an invariant product ($ab = cx$).

Fostering Attendance to Meaning. The hammer-and-nail phenomenon is more likely to manifest among students with superficial understanding. For example, students might use a proportion, $a/b = c/x$, to solve a missing-value problem without knowing what in the context of the problem situation each ratio refers to and why the two ratios should be equal to each other. Such students are said to be reasoning in a *non-referential symbolic* manner—"operating on symbols as if they possess a life of their own" (Harel, Fuller, & Rabin, 2009, p. 2008). This way of thinking often leads to errors. It is therefore important for students to attend to the meanings of the symbols and numbers they encounter in a problem. To foster attendance to meaning in the context of proportional reasoning, students should be challenged to explain what the ratio a/b represents in the context of the problem situation and why it should be equal to the other ratio, c/x. When students develop the habit of attending to meaning, they are more likely to engage in quantitative reasoning and less likely to respond to a problem impulsively.

Concluding Remarks

This paper uses the hammer-and-nail metaphor to highlight the rigidity and impulsiveness that can be found in student mathematical behaviour. The hammer-and-nail phenomenon can be attributed to two plausible causes: the way the human mind works and the way mathematics is traditionally taught in school. To deal with the first factor, we can help students become cognizant of their own impulsive tendency, be more metacognitive, and develop the habit of utilizing System 2 instead of simply letting System 1 dominate. To address the second factor, classroom instruction and assessments should place greater emphasis on relational understanding, analysing, sense-making, and attending to meaning of symbols.

The hammer-and-nail phenomenon deserves the attention of the mathematics education community because it reminds teachers that many students are learning mathematics without engaging in mathematical reasoning and sense making. Awareness of this phenomenon can also help teachers become more sensitive of their own teaching so as to not foster students' impulsive tendencies. Teacher educators can use the hammer-and-nail phenomenon to highlight the futility of instrumental understanding and to motivate teachers to teach in a manner that fosters relational understanding and analytical reasoning.

References

Ben-Zeev, T., & Star, J.R. (2001). Spurious correlations in mathematical thinking. *Cognition and Instruction, 19*(3), 253-275.

Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 159–178). New York: Macmillan.

Davis, R.B. (1989). The culture of mathematics and the culture of schools, *Journal of Mathematical Behaviour, 8*, 143-160.

Harel, G., Fuller, E. & Rabin, J. (2008). Attention to meaning by algebra teachers. *Journal of Mathematical Behaviour, 27*, 116-127.

Hinsley, D., Hayes, J.R., & Simon, H.A. (1977). From words to equations. In P. Carpenter & M. Just (Eds.), *Cognitive processes in comprehension* (pp. 89-106). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kagan, J. (1965). Impulsive and reflective children: Significance of conceptual tempo. In J. D. Krumboltz (Ed.), *Learning and the educational process* (pp. 133–161). Chicago: Rand McNally.

Kagan, J., Rosman, B., Day, D., Albert, J., & Phillips, W. (1964). Information processing and the child: Significance of analytic and reflective attitudes. *Psychological Monographs, 78* (No. 1, whole number 578).

Kaplan, A. (1964). *The conduct of inquiry: Methodology for behavioural science*. San Francisco: Chandler Publishing Co.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Reston, VA: NCTM.

Lim, K. H. (2009). Burning the candle at just one end: Using nonproportional examples helps students determine when proportional strategies apply. *Mathematics Teaching in the Middle School, 14,* 492–500.

Lim, K. H., & Morera, O. (2010). Addressing impulsive disposition: Using non-proportional problems to overcome overgeneralization of proportionality. *Online proceedings for the 13th SIGMAA on Research in Undergraduate Mathematics Education Conference*. Raleigh, NC.

Lim, K. H., Morera, O., & Tchoshanov, M. (2009). Assessing problem-solving dispositions: Likelihood-to-act survey. In S.L. Swars, D.W. Stinson & S. Lemons-Smith (Eds.). *Proceedings of the 31st annual conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 700-708). Atlanta: Georgia: PME-NA.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics, 38,* 135–161.

Maslow, A. H. (1966). *The psychology of science: A reconnaissance*. New York: Harper Collins.

Messer, S. B., & Brodzinsky, D. M. (1981). Three-year stability of reflection–impulsivity in young adolescents. *Developmental Psychology, 17,* 848-850.

Monteiro, C. (2003). Prospective elementary teachers' misunderstandings in solving ratio and proportion problems. In N. A. Pateman, B. J. Dougherty and J. T. Zilliox (Eds.) *Proceedings of the 27th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 317–234). Hawaii, USA: PME.

National Research Council. (2000). *How people learn: Brain, mind, experience, and school*. Washington, DC: National Academic Press.

Nietfeld, J., & Bosma, A. (2003). Examining the self-regulation of impulsive and reflective response styles on academic tasks. *Journal of Research in Personality, 37,* 118–140.

Reber, A. S. (1993). *Implicit learning and tacit knowledge*. Cambridge, MA: MIT Press.

Riding, R., & Indra, C. (1991). Cognitive styles: An overview and integration. *Educational Psychology, 11,* 193–215.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 7,* 20-26.

Sloman, S. A. (1996). The empirical case for two systems of reasoning. *Psychological Bulletin, 119,* 3-22.

Stanovich, K.E., & West, R.F. (2000). Individual differences in reasoning: Implications for the rationality debate. *Behavioural and Brain Sciences, 23,* 645-726.

Stigler, J. W., & Hiebert, J. (1999). *The thinking gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25,* 165–208.

Tirosh, D., & Stavy, S. (1999). Intuitive rules: A way to explain and predict students' reasoning. *Educational Studies in Mathematics, 38,* 51-66.

Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Vershaffel, L. (2004). Remedying secondary school students' illusion of linearity: A teaching experiment aiming at conceptual change. *Learning and Instruction, 14,* 485–501.

Watson, A., & Mason, J. (2007). Taken-as-shared: A review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education, 10,* 205–215.